

Evolution and extinction of families in cellular automata

György Szabó and István Borsos

Research Institute for Materials Science, H-1525 Budapest, P. O. Box 49, Hungary

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In a large class of cellular automata a unique “parent” particle of the previous state can be assigned to each particle of the present state. This allows us to define families and study their evolution and extinction in one-dimensional cellular automata. The size density of families is found to tend towards a universal function for large times. The evolution of the average family size, proportional to \sqrt{t} , is strongly related to the ordering mechanism found by Grassberger [Phys. Rev. A **28**, 3666 (1983)].

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Cellular automata (CA) theory is applied in a great variety of physical, chemical, and biological systems [1]. The simplest such automaton is defined on a regular one-dimensional lattice with two states per lattice point at a discrete time t . We can say that site i is empty or occupied by a particle. The discrete time evolution is determined by a rule dependent on the local neighborhood. For example, in a set of one-dimensional models $S_i(t+1)$ depends on three sites, i.e., $S_i(t+1) = F[S_{i-1}(t), S_i(t), S_{i+1}(t)]$. In this case there can be 2^{2^3} rules numbered from 0 to 255. Some general features of these models are clarified by Wolfram [1] and Martin *et al.* [2]. For a given class of CA the particles created by the transition rule may originate from a single “parent.” This feature permits the particles to be grouped into families, a family being the set of all particles originating from the same particle of an earlier state.

Henceforth we restrict ourselves to investigating the one-dimensional CA with Rule 18 [1] on a lattice of L sites when $L \rightarrow \infty$. For simplicity, the families will be denoted by positive integers, i.e., $S_i(t) = 1, 2, \dots$ for occupied sites and $S_i(t) = 0$ for empty sites. Evolution according to this notation is expressed as

$$S_i(t+1) = \begin{cases} S_{i-1}(t) & \text{for } S_i(t) = S_{i+1}(t) = 0 \\ S_{i+1}(t) & \text{for } S_{i-1}(t) = S_i(t) = 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

This rule reproduces the traditional Rule 18 [1] when all the particles belong to the same family, i.e., $S_i(t) = 0$ or 1.

If this system is started from a single-particle state (at $t = 0$) then the configurations at successive time steps form the lines of Pascal’s triangle modulo 2 [1]. In this case the size of the family grows linearly with time, i.e., $s = 2t + 1$.

If the initial state contains more than one ancestor, then the growing families will meet sooner or later. According to (1) this contact prevents the expansion of both families at this point. Occasionally, these processes may result in the extinction of a family and the neighboring ones will occupy the deserted territory. As a consequence, the number of families decreases and their average size increases with time.

In our analysis we will concentrate on systems started

from a random initial state. In order to visualize the evolution and extinction of families we display a part of the configuration:

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0 0 1 0 0 2 0 0 0 3 0 4 0 0 0 0 5 0 0 0 6 0 0 0 7 0 0 0 0 0
0 1 0 1 2 0 2 0 3 0 0 0 4 0 0 5 0 5 0 6 0 6 0 7 0 7 0 0 0 8
1 0 0 0 0 0 0 0 3 0 4 0 4 5 0 0 0 0 0 0 0 0 0 0 0 7 0 8 0
0 1 0 0 0 0 0 0 3 0 0 0 0 0 0 5 0 0 0 0 0 0 0 0 0 7 0 0 0 8
1 0 1 0 0 0 0 3 0 3 0 0 0 0 5 0 5 0 0 0 0 0 0 0 7 0 7 0 8 0
0 0 0 1 0 0 3 0 0 0 3 0 0 5 0 0 0 5 0 0 0 0 0 7 0 0 0 0 0 8
0 0 1 0 1 3 0 3 0 3 0 3 5 0 5 0 5 0 5 0 0 0 7 0 7 0 0 0 8 0
0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 5 0 7 0 0 0 7 0 8 0 8
1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 5 0 0 0 7 0 7 0 0 0 0 0
0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 5 0 5 0 7 0 0 0 7 0 0 0 0

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The first line illustrates a random configuration at $t = 0$, where 0 indicates an empty site and the numbers from 1 to 8 refer to the family names of the particles. In the initial state each particle represents an ancestor. The particle configurations at successive time steps are shown line by line. During this process the particles inherit their family name as denoted.

Some features are preserved in the family. For example, if an ancestor is at an even site for $t = 0$, then its descendants reside at odd sites for odd times and even sites for even times. This type of “ordered” region is called phase *A*. There exists another phase (*B*) in which the particles reside at odd (even) sites for even (odd) times. Each family belongs to one of the ordered phases. Neighboring families may form a larger ordered domain of phase *A* or *B*. If the phase of consecutive families is randomly *A* or *B* then the number of families is twice the number of domains when $L \rightarrow \infty$. Grassberger has shown that the average density of domains decreases as $1/\sqrt{4\pi t}$ for $t \rightarrow \infty$ [3]. Consequently, the density of families should be equal to $1/\sqrt{\pi t}$ for $t \rightarrow \infty$. To demonstrate it we have performed simulations on a lattice ($L = 30\,000$) with a periodic boundary condition. In the random initial state we have chosen the density of particles (and family ancestors) to be equivalent to the stationary value ($c = 1/4$). In Fig. 1 the closed boxes and circles demonstrate the density of families and do-

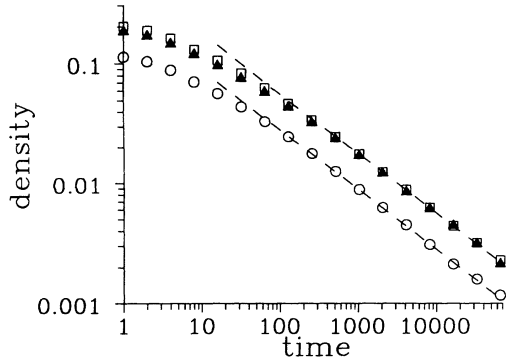


FIG. 1. Log-log plot of average family (\square) and domain (\circ) densities vs time for simulations starting from a random configuration. The filled triangles indicate family densities when starting from a monodomain state.

mains as a function of time, and dashed lines represent theoretical results.

The above simulation was repeated by choosing a monodomain (stationary) initial state. In other words, the particles (ancestors) are positioned randomly ($c = 1/4$) at even sites for $t = 0$. The open circles in Fig. 1 clearly show that the density of families tends towards the same asymptotic behavior as above while the number of domains remains 1 (corresponding to zero density). This observation has inspired us to study the evolution and extinction of families in more detail.

The boundary between the neighboring families is defined as the center of the empty region separating them. This boundary is equivalent to a kink (or antikink) [3] if the families belong to different phases (i.e., the no-man's land consists of an even number of empty sites). Its motion has already been investigated by several authors [3–5]. The kink walks randomly with a diffusion constant $D = 1/2$. It is easy to see that the motion of the boundary remains unchanged when inserting an empty site at the center of the boundary region. Consequently, the motion of the boundary is independent of the phase of families concerned.

For large times this system tends towards a stationary state built up from a random series of (10) [or (01)] and (00) blocks appearing with the same probability. In this limit, straightforward calculation gives that the boundary jumps k sites with a probability

$$p(k) = \frac{1}{12} \left(\frac{1}{2}\right)^{|k|} + \frac{3}{4} \delta_{k,0}, \quad (2)$$

where $\delta_{k,j}$ is the Kronecker symbol. This jump rate is symmetric, that is, $p(-k) = p(k)$ and

$$\sum_{-\infty}^{\infty} k^2 p(k) = 2D = 1. \quad (3)$$

If $x(t)$ is the position of a boundary (or kink) then the above expression leads to $\langle \delta x^2 \rangle = t$ because the subsequent jumps are independent of each other.

In this system the motion of kinks and antikinks may be considered as an annihilating random walk [6]. The

annihilation of a kink-antikink pair is accompanied by the disappearance of the domain between them and the union of the neighboring domains. This process results in a complicated domain growth mechanism. Similar processes have already been investigated by Sekimoto [9] and Alexandrowicz [10] by studying the time evolution of size density functions (further references therein).

The situation is different for families. Here the boundaries undergo a coalescing random walk [6–8] and simultaneously the corresponding family disappears without leaving a trace. In comparison to the previous case, the growth mechanism is much simpler now because of the absence of family aggregation.

Introducing a continuous size variable s we define the size density function $f(s, t)$ of families such that the number of families with sizes $(s, s + ds)$ is $f(s, t) ds$ normed to the length of the system. Evidently, $f(s, t) > 0$, for $s > 0$ and it satisfies the following condition:

$$\int_0^{\infty} s f(s, t) ds = 1. \quad (4)$$

For sufficiently large family sizes the time variation of $f(s, t)$ is determined by the random walk of the boundaries, that is,

$$\partial_t f(s, t) = \sum_{k_1, k_2} p(k_1) p(k_2) [f(s - k_1 + k_2, t) - f(s, t)], \quad (5)$$

where ∂_t refers to derivation with respect to time. Assuming that $f(s, t)$ is a smooth function of s , $f(s - k_1 + k_2, t)$ can be replaced by its Taylor series with respect to $(k_2 - k_1)$. The leading term yields a differential equation analogous to the one-dimensional Fick's law,

$$\partial_t f(s, t) = \partial_{ss} f(s, t), \quad (6)$$

where ∂_{ss} denotes the second derivative with respect to s and the unit value of the "diffusion coefficient" comes from Eq. (3). In fact, this equation is valid if s is much larger than the typical boundary jump comparable to 1 [see Eq. (3)]. For the sake of simplicity we extend the validity of this equation to arbitrary $s > 0$.

The disappearance of families with zero extension represents a boundary condition, namely,

$$f(s = 0, t) = 0. \quad (7)$$

Equation (6) with this boundary condition provides that the normalization expressed by (4) remains valid if $s \partial_s f(s, t)$ and $f(s, t)$ tend to zero for large s .

The solution of Eq. (6) with the boundary condition (7) is equivalent to a diffusion phenomenon with antisymmetric initial state [$f(-s, t = 0) = -f(s, t = 0)$]. The general solution is given as

$$f(s, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(q) e^{-q^2 t + i q s} dq, \quad (8)$$

where $a(q)$ is an odd imaginary function of q and its value is determined by the initial distribution for $t = 0$. The

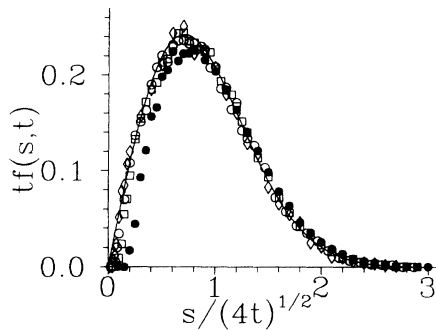


FIG. 2. Scaled size density function for $t = 100$ (\bullet), 400 (\square), 900 (\circ), and 1600 (\diamond). The solid line represents the theoretical result.

function $a(q)$ is prescribed by Eq. (4) for small q , namely, $a(q) \simeq -2iq$. As a consequence, we can evaluate $f(s, t)$ for large times because the exponential factor suppresses the contributions belonging to large q . Thus, the asymptotic behavior can be expressed as

$$f(s, t) \simeq \frac{s}{\sqrt{4\pi t^3}} e^{-s^2/4t}, \quad (9)$$

or

$$tf(s, t) = \frac{1}{\sqrt{\pi}} ye^{-y^2}, \quad (10)$$

where $y = s/\sqrt{4t}$. The same result has been obtained by Bramson and Griffeath [8] using more rigorous techniques. This agreement confirms the validity of the assumption made in the derivation of Eq. (6). This universal behavior has also been checked by simulations on a lattice of 30 000 sites with a random initial configuration of stationary concentration ($c = 1/4$). The results plotted in Fig. 2 have been obtained by averaging over 100 runs. Slight deviation from the theory (dashed line) can be observed for small sizes and short times. These deviations come from the fact that small families dominate

the system at the beginning and we have not excluded those jumps which would result in negative family sizes in Eq. (6). This former simplification does not affect the results for large times.

With the knowledge of $f(s, t)$ we can easily determine the average density of families, i.e.,

$$\varrho(t) = \int_0^\infty f(s, t) ds \simeq \frac{1}{\sqrt{\pi t}} \quad (11)$$

for large times. This result agrees with the expectations sketched above and shown in Fig. 1.

Although the present analysis is restricted to the CA with Rule 18, the concept of family may successfully be applied for investigating other CA (e.g., Rule 22) mentioned by Grassberger [3].

It is worth mentioning that the growth of families is analogous to a particular problem of the gambler's ruin [11] where each gambler sitting around a table can win (or lose) a certain sum from his neighboring adversaries. A gambler's capital is equivalent to the family size and gamblers losing their capital leave the game. In this case, $f(s, t)$ gives the time dependence of capital distribution among a large number of players.

In summary, we have generalized the one-dimensional CA with Rule 18 by distinguishing the particles and recording their descendants. This generalization has allowed us to introduce the concept of a family whose growth is strongly related to the ordering mechanism found by Grassberger. The continuous description of the time evolution of the family size density function suggests a universal behavior for large times independent of the initial state. Simulations have confirmed this theoretical prediction. The present concept of a family seems to play a significant role when analyzing domain dynamics for other systems.

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